Some of the most influential approaches to development, based on the work of Piaget (1952), stress the relationship of the concept of conservation to mathematical knowledge and ability. Students have a concept of conservation if they understand, for example, that lengths of objects do not change when they are moved or that the amount of a liquid does not change when it is poured into a container of a different size or shape. These general rules of development help in understanding mathematical concepts.

In addition to general development, it also helps to look at mathematical development specifically. In what order do children and adolescents acquire concepts and operations? How does a student's approach to arithmetic change with age? Many authors have discussed these transformations and presented developmental theories of children's understanding of mathematical concepts (Carpenter & Moser, 1982; Geary, 1994).

Contemporary theory about mathematical development holds that students match new information with what they already know. Most of what they know about mathematics is intuitive and is often called informal mathematical knowledge (Geary, 1994; Ginsburg, 1977, 1997). Formal mathematical knowledge, in contrast, is the system of symbols, concepts, procedures, and so forth that form the content of mathematics instruction.

Normally developing young children usually come to school with some informal knowledge. In fact, infants as young as 4 or 5 months of age can discriminate between arrays of two and three objects. By about 18 months, they appear to understand something about numerical order (3 comes after 2). Thus, throughout the preschool years, children learn much about counting and numbers. Twin studies reveal that some mathematics skills are innate (Allardé, Defries, Light, & Pennington, 1997) but that they are also influenced by the environment. For example, children and parents often play counting games such as “One, two, buckle my shoe.” Shannon’s father recalls his daughter’s difficulty with this nursery rhyme:

**Shannon** enjoyed hearing nursery rhymes when she was a preschooler, but she had trouble catching on when she tried reciting them on her own. She seemed to have particular trouble with rhymes that involved numbers. I remember demonstrating the rhythmic pattern for her. I’d say, “Listen, Shannon...three, four, shut the door; five, six, pick up sticks.” She seemed to try, but she seemed confused. She often mixed up the phrases.

By age 3 or 4 years, many children actually use number names to identify the number of objects in a group, although they can do so only when the group has a few items (Geary, 1994).

Researchers (e.g., Gersten & Chard, 1999) use the term number sense to describe a conceptual structure critical for mathematics learning. Number sense refers to a child’s fluidity and flexibility with numbers, the sense of what numbers mean, and an ability to perform mental mathematics and to look at the world and make comparisons (Gersten & Chard, 1999, pp. 19-20). Most children develop this sense before kindergarten through informal interactions with family members. Number sense leads to greater facility with math information and the ability to solve arithmetic computations (Gersten & Chard, 1999). The Current Trends and Issues box above provides insight into how phonological processing skills and number sense might be related to our understanding of math disabilities.

---

**Shannon**

Shannon enjoyed hearing nursery rhymes when she was a preschooler, but she had trouble catching on when she tried reciting them on her own. She seemed to have particular trouble with rhymes that involved numbers. I remember demonstrating the rhythmic pattern for her. I’d say, “Listen, Shannon...three, four, shut the door; five, six, pick up sticks.” She seemed to try, but she seemed confused. She often mixed up the phrases.

By age 3 or 4 years, many children actually use number names to identify the number of objects in a group, although they can do so only when the group has a few items (Geary, 1994).
When they arrive at school, many children use their number sense to solve simple arithmetic problems (Rensick, 1983; Rensick & Ford, 1981). These skills are probably based on a rudimentary understanding of the relationships between counting, numbers, and numerals. The concepts can be represented as a number line, shown in Figure 14.1. The concept of a mental number line holds that each number is conceptually linked to the next higher one in the way that it might be after extensive practice with counting. Each number is also linked to a concept of the number of "things" it represents (as represented by the drawings of dots). Thus, the mental number line represents children's understanding of the fundamental concepts that numbers occur in order and that numerals are used to represent numbers of objects. This concept is further discussed in Focus on Mathematics on pages 485-487. As young children grow more sophisticated, they learn other concepts to go with the mental number line. For example, they learn the related concepts of equality (in the mathematical sense) and correspondence.

Very young children typically have simple addition skills. By the time they begin to progress through the primary grades, children show further development of addition concepts. Regardless of culture, addition skills appear to be based on counting (Geary, 1994). Initially, children employ very simple strategies to add two sets or groups. For example, they count out the objects for one group ("one, two, three"), count out the objects for the second group ("one, two, three, four"), and then physically move the groups together. Once the groups are combined, they count the total ("One, two, three, four, five, six, seven ... seven eggs! Don't wanna drop these!").

In addition to the concept of a number line, primary-school children learn to think about numbers as wholes that are composed of parts. For example, the number 7 may also be thought of as being composed of the numbers 4 and 3, 5 and 2, and 6 and 1. Understanding the part-whole concept of numbers allows school-age children to interpret and solve more sophisticated problems than preschoolers. Some instructional programs—for example, Connecting Math Concepts (Engelmann, Carnine, Engelmann, & Kelly, 1991)—base instruction on simple computation and even on problem solving or mastery of part-whole relationships.

As they develop greater facility with mathematics, children gradually adopt different strategies. For example, they progress from the counting of manipulatives to more and more efficient strategies, which require fewer cognitive actions. Table 14.1 (page 456) shows the usual developmental pattern of strategies through which most children go as they become more efficient with addition. Similar patterns also emerge for other forms of computation, such as subtraction (Geary, 1994).

Unfortunately, many educational methods for teaching arithmetic do not conform to the normal development of computational skill. The sequence of development shown in Table 14.1 illustrates this problem. Many education methods encourage young children to work with manipulatives when learning addition. First, they are shown the "counting manipulatives" strategy illustrated in Table 14.1. This system of instruction works well for students who have a lot of knowledge of mathematical concepts and competency with numbers. For students who lack those skills, skipping from a counting-manipulatives to a fact-retrieval strategy likely will cause problems.

Another major developmental step in understanding numbers is learning the working of the decimal, or base-10, system. At first, children are taught to treat two-digit numbers (e.g., 17) as composed of two parts (10 + 7); then they are instructed to extend this to other numbers, with the requirement that one of the parts must be a multiple of 10 (e.g., 43 is 40 and 3). Handling numbers in this manner (later extended to 100s, 1,000s, and so forth) is sometimes called the concept of place value.

Development of this concept allows students to perform more complex computations "in their heads" by using strategies related to those they acquired earlier. For example, 147 + 265 is composed of (100 + 200), (40 + 60), and (7 + 5); these resolve to (300 + 100) + 12, which is 412.

As shown in Table 14.1, older students who solve more difficult problems also gradually adopt more efficient strategies. Their informal knowledge about more sophisticated topics gradually changes as they learn formal mathematics. For example, most young children understand the meaning of "half" when it refers to a cookie. However, as they learn more, they understand that the concept of "half" does not stand only for physical relationships. They must learn that the idea of "half" can be applied to "half of the boys whom I know" (and that it does not require that each boy is cut into two pieces).

More sophisticated concepts in mathematics develop later. Computational skill and conceptual knowledge continue to change with age. Many of the advances in comprehension have to do with acquiring and using more sophisticated strategies. Students who do not advance beyond using elementary strategies (e.g., counting manipulatives or counting fingers, as illustrated in Table 14.1) have greater trouble with advanced concepts. For students with learning disabilities, acquiring advanced concepts is a particularly important problem.
### Table 14.1: Commonly Used Addition Strategies

<table>
<thead>
<tr>
<th>STRATEGY</th>
<th>DESCRIPTION</th>
<th>EXAMPLE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simple Addition</td>
<td>The problem's augend and addend are represented by objects. The objects are then counted, starting from 1.</td>
<td>To solve $2 + 3$, two blocks are counted out, then three blocks are counted out, and finally all five blocks are counted.</td>
</tr>
<tr>
<td>Counting manipulatives</td>
<td>The problem's augend and addend are represented by fingers. The fingers are then counted, usually starting from 1.</td>
<td>To solve $2 + 3$, two fingers are lifted on the left hand, and three fingers are then lifted on the right hand. The child then moves each finger in succession as he or she counts them.</td>
</tr>
<tr>
<td>Counting fingers</td>
<td>The child counts the augend and addend in succession starting from 1.</td>
<td>To solve $2 + 3$, the child counts &quot;1, 2, 3, 4, 5; the answer is 5.&quot;</td>
</tr>
<tr>
<td>Counting on first (min)</td>
<td>The child states the value of the augend and then counts a number of times equal to the value of the addend.</td>
<td>To solve $2 + 3$, the child counts &quot;2, 3, 4, 5; the answer is 5.&quot;</td>
</tr>
<tr>
<td>Counting on larger (min)</td>
<td>The child states the value of the larger addend and then counts a number of times equal to the value of the smaller addend.</td>
<td>To solve $2 + 3$, the child counts &quot;3, 4, 5; the answer is 5.&quot;</td>
</tr>
<tr>
<td>Verbal counting</td>
<td>One of the addends is decomposed into two smaller numbers, so that one of these numbers can be added to the other to produce a sum of 10. The remaining number is then added to 10.</td>
<td>To solve $8 + 7$, Step 1: 7 = 5 + 2 Step 2: 8 + 2 = 10 Step 3: 10 + 5 = 15</td>
</tr>
<tr>
<td>Derived facts (decomposition)</td>
<td>The problem is solved by retrieving columnwise sums.</td>
<td>To solve $27 + 38$, Step 1: 7 + 8 = 15 Step 2: Note trade (carry) Step 3: 3 + 2 + 3 = 8 Step 4: 5 + 1 (from trade) = 6 Step 5: 6 + 5 = 11 from ones column to 5 from tens column to produce 65</td>
</tr>
</tbody>
</table>


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### What Problems in Mathematics Do Students Experience?

Students with learning disabilities experience a wide variety of problems in mathematics (Strang & Rourke, 1985; Geary, Hamson, & Hoard, 2000). Thus, just as with reading or writing disabilities, any effort to characterize "the math disabled child" is a mistake; students who have mathematics disabilities are too heterogeneous to constitute a type. Dyscalculia is the most widely used term for disabilities in mathematics. In general, dyscalculia means a severe or complete inability to calculate. Studies indicate that "between 6 and 7% of school-age children show persistent, grade-to-grade, difficulties in learning some aspects of arithmetic or related areas" (Geary, 1999, p. 2). More than half of students with learning disabilities have individual education program goals in math (Kavale & Reece as cited in Fuchs & Fuchs, 2001). Students with learning disabilities who are in the third and fourth grades often score at a first-grade level in arithmetic. Things do not get better later; high school students often score at about the fifth grade level (Cawley & Miller, 1989), and difficulties continue in postsecondary educational settings (Stawarz & Miller, 2001). Clearly, students with such deficits deserve special education.

Some students with learning disabilities have problems in both mathematics and reading (Robinson et al., 2002). Others have problems in only one area (Lewis, Hitch, & Walker, 1994). The performance of students who have only mathematics problems differs from that of those who have problems in both reading and mathematics. For example, when tests are timed, students with only mathematics problems make about as many mistakes on simple story and number-fact problems as their peers who have both mathematics and reading problems. However, when tests are untimed, those with only mathematics problems make fewer mistakes and get just about as many problems correct as nondisabled students (Jordon & Montani, 1997).

### Problems in Cognitive Development

Authorities in learning disabilities have suggested many types of problems that may be associated with disabilities in learning mathematics. Many of these difficulties are not directly related to arithmetic performance but fall into the categories of developmental problems and information-processing disorders. Research in cognitive science indicates that students must have an understanding of mathematical facts (i.e., declarative knowledge), rules and procedures (i.e., procedural knowledge), and relationships (i.e., conceptual knowledge) in order to develop mathematical literacy. Students who have difficulty with math, for example, typically approach word problems in a mechanical fashion (e.g., by looking for key words) rather than attempting to understand the problem. In doing so, they often use irrelevant information and are unable to determine if their answer makes sense.
It seems like Shannon's always had problems with math. She just can't grasp the concepts. I remember she did problems like '[a] bus holds 30 students, how many buses would be needed to take 180 students on a field trip?' With the help she did the calculations right—too divided by 30 is 6.2. But then she'd say that 3.2 is the answer, I'd try to explain that they wouldn't be able to take one third of a bus along, that they'd need 4.

As Goldman and colleagues (1997) note, mathematical competence involves far more than fluent retrieval of basic math facts and the execution of computational procedures. To be sure, learners must possess a sufficient and efficient knowledge base from which to draw information. The retrieval of facts and basic procedural steps to solve a problem depends on recognizing their appropriate use—a skill that requires the coordination of relationships among declarative, procedural, and conceptual knowledge. (p. 202)

Problems in Arithmetic Performance

Some of the mathematical difficulties that students with learning disabilities have are directly associated with performance of arithmetic tasks. For example, students with dyscalculia often have problems with such skills as (1) writing numerals and mathematical symbols correctly, (2) recalling the meanings of symbols and the answers to basic facts, (3) counting, and (4) following the steps in a strategy for solving multiplication problems (Glennen & Cruickshank, 1981).

Performance on Basic Arithmetic Tasks

Students rarely make random mistakes in answering arithmetic problems. Classic studies have documented that the errors are usually systematic and indicate that students are consistently applying a mistaken strategy to solve the problems (Cox, 1975; Ginsberg, 1977; Laskford, 1972). The analysis of computational errors has a long history (Buswell & John, 1926). Extensive analyses of "bugs," or errors in computation, have also been made by educators and psychologists interested in children's thinking during solution of arithmetic problems (Ashlock, 1994; Woodward & Howard, 1994; Young & O'Shea, 1981).

Cognitive research on mathematical difficulties reveals that such students have deficits in fact retrieval (Garnett & Flehinger, 1983; Geary, 1994; Geary, Hoard, & Hamson, 1999). They make more mistakes in giving simple answers in various areas of arithmetic and sometimes recall facts more slowly than their peers. Such fact retrieval problems are probably related to deficits in working memory (see Chapter 8).

Students also make mistakes in applying strategies or procedures (Geary, 1994; Jordan & Montani, 1997). They may not only choose inefficient strategies, but also poorly use those they choose. For example, students with problems are more likely to depend on the counting-all strategy (see Table 14.1), even though it is less efficient than strategies developed later. When they revert to using this strategy, they may also make mistakes in counting, leading to wrong answers.

Difficulties with Story Problems

Difficulties with computation will have obvious effects on whether students can solve story problems correctly. Problems reading the stories will affect performance, too, but students' difficulties with these kinds of tasks are more complex than one would predict because of reading or computation deficits alone.

Certain aspects of story problems make them difficult for many students. For example, story problems given in reverse order and beginning with the missing number (e.g., ones for which an equation might be written in this way: 7 = 5 + x) are considerably harder than other story problem arrangements (Rosenthal & Resnick, 1974). The difficulties unique to solving story problem arrangements by students with learning disabilities reveal that these students' performances are adversely affected by such features of story problems as (1) presence of extraneous information, (2) use of complex syntactic structures, (3) change of number and type of noun used, and (4) use of verbs such as "purchased" or "bought" rather than "was given" (Blankenheim & Lovitt, 1976; Entjeda, DiPipi, & Perron-Jones, 2002; Parmar, Cawley, & Fraenzi, 1996; Trenholme, Larsen, & Parker, 1973). The implica tion of these findings is not that teachers of students with learning disabilities should avoid assigning problems with these features. Instead, they should teach their students how to solve them by applying strategies such as Solve It (Montague, Wariger, & Morgan, 2000) and STAR (Maccini & Hughes, 2000). See the Case Connections box on pages 460–461 for an example of how Shannon uses the STAR strategy.

Students with learning disabilities are vulnerable to inadequate instructional programs and practices. For example, if a program for teaching fractions does not use both proper fractions and improper fractions when demonstrating how to work with fractions, students with learning disabilities are far more likely to make mistakes on problems involving improper fractions (Kelly, Gersten, & Carnine, 1990). Fractions is not the only area in which problems are likely to occur. Figure 14.2 (page 462) illustrates the range of problem areas in which students are likely to have trouble. If instructional practices, including both curriculum and teaching behavior, do not correct for problems in these areas, students with learning disabilities will be most likely to suffer. "Low-achieving students are often casualties of educational systems that are too much and teach too little. Difficulties are related to learners' fragile preskills and the subsequent failure of the curriculum to explicitly address those skills" (Kame'enui & Simmons, 1990, p. 393).
Applying a Problem-Solving Strategy: Shannon Ireland Explains

Teachers have used the STAR strategy (see Figure A) successfully with secondary school students with learning disabilities (Maccini & Hughes, 2000). Students are taught to read the problem, self-question, and write down facts. Then they learn to translate the words into an equation by progressing through three phases of instruction. In the first phase (i.e., concrete application), students represent the problem using objects or a number line. In the second phase (i.e., concrete application), they represent the problem by drawing a picture. In the third phase (i.e., abstract application), they use numerical symbols. With the help of her teacher, Shannon has progressed through the first two phases. She is now in the third phase, where she is able to represent equations using numerical symbols. Here, Shannon explains how she uses the STAR strategy:

"I've learned the STAR strategy. I use it for word problems in algebra, like this one: 'Monday, the temperature was -8°F. By Wednesday, the temperature had risen 15°F. What was the temperature on Wednesday?' First, I search the problem. That means I read it over carefully and figure out what facts I know and what I need to find. I know that the temperature was -8°F on Monday, and it rose 15°F by Wednesday. I need to figure out what the temperature was on Wednesday. Now, I have to translate the words into an equation. I see, 'Monday, the temperature was -8°F. By Wednesday, the temperature had risen 15°F.' What was the temperature on Wednesday? So, the equation would be -8 + 15 = x. Is that right? -8 on Monday plus 15, that's how much the temperature rose, equals x, the temperature on Wednesday. 'Yes,' that looks right. Okay, so now I answer: I have different signs, so I subtract and keep the sign of the bigger number; 15 minus 8 is 7, because since the sign of the bigger number, 15, is positive. The temperature on Wednesday was 7°F. Now, I review. The problem was: 'Monday, the temperature was -8°F. By Wednesday, the temperature had risen 15°F. What was the temperature on Wednesday?' Does my answer make sense? I think so, but I'm not sure. Let me check. Negative 8 plus positive 15 is 7. Because when you're adding, and you have different signs, you subtract and take the sign of the bigger number minus positive 7. That's right. There's how I use the STAR strategy!"

How Are Mathematics Abilities Assessed?

Mathematics learning problems are assessed in the same way as deficits in other areas of academic learning. Teachers may make referrals because students appear to be having difficulties, and students may be administered screening tests to determine whether further assessment is needed. Another type of assessment includes testing used to guide program planning. This form of assessment is designed to help determine what specific arithmetic learning problems students have and what kind of educational program will be needed to remedy them. When remediation is under way, assessment continues in the form of progress monitoring (Bryant & Rivera, 1997).
FIGURE 14.3 Predictable Problem in Mathematics' Skill Domain for Students with Learning Disabilities

Predictable problems in mathematics skill domain

- Math-related language
- Numeration
- Computation
- Problem solving

Basic concept knowledge
- Rational counting
- Basic facts
- Complex grammar
- Problem with missing values
- Vocabulary and verbal cues

Classification knowledge
- Counting from a number
- Regrouping
- Story problem structure

Numerical identification
- Zeros in the quotient
- Multiple operations

Reading and writing numbers
- Column alignment
- Extraneous information

Expanded notation
- Complex computational algorithms
- Functional application

Error patterns


Unfortunately, most of the work on assessing mathematical competence of students with learning disabilities focuses almost exclusively on computation. Too little work has been done on assessing mathematical concepts. Because knowledge of underlying concepts facilitates mastery of arithmetic computation, students' understanding of concepts is an appropriate area for assessment. Similarly, students' strategies in solving problems are worthy of assessment (Ginsburg, 1991, 1997).

Achievement Tests

To identify students who might need additional services, schools may screen large groups of students in mathematics. Administration of a screening test helps educators decide whether students should be tested further to determine if they have a mathematics learning problem or, if one is suspected, to confirm it.

Screening usually consists of administering a norm-referenced test to compare the referred student to normally performing age- and grade-mates. When a student is generally behind age- or grade-mates on arithmetic screening tests, an arithmetic learning disability is suspected. Commonly used screening tests include most of the general achievement batteries: the California Achievement Test—5th Edition (Teige & Clark, 1992), the Iowa Test of Basic Skills (Kieronymus, Lindquist, & Hoover, 1978), the Metropolitan Achievement Test—7th Edition (Frescott, Balow, Hogan, & Farr, 1992), and the Stanford Achievement Test—8th Edition (Gardner, Rudman, Kaufman, & Merwin, 1988). However, individually administered achievement batteries such as the Peabody Individual Achievement Test—Revised (Markwardt, 1989) and the Woodcock-Johnson III (Woodcock, McGrew, & Mather, 2001), as well as some tests devoted specifically to diagnosing arithmetic and mathematics (discussed in the following section on formal diagnostic testing), probably are the instruments most commonly used when identifying students.

Some screening instruments may provide preliminary diagnostic information. For example, the Metropolitan Achievement Tests (Frescott et al., 1992) contain two subsets related to mathematics: one that assesses computation skills and one that assesses knowledge of concepts. If a student performs poorly on one of these but not on the other, the difference probably reflects something about his or her difficulties with mathematics. Some authors (e.g., Trembly, Caponipo, & Gaffrey, 1980) suggest programming based on common achievement tests such as the PIAT and the Wide Range Achievement Test—3 (Wilkinson, 1993). However, when planning instructional programs, teachers need much more fine-grained assessment measures than tests of this sort offer.

Formal Diagnostic Testing

Diagnostic tests should allow the teacher to determine in which areas of mathematics a student is having difficulties. Table 14.2 (page 464) identifies some of these areas and gives examples of them. Diagnostic tests sample from some or all of these.

Key Math Revised: A Diagnostic Inventory of Essential Mathematics (Connolly, 1998) is a very widely used diagnostic instrument for grades kindergarten through 8. Its 14 subsets are arranged into three general areas: content, operations, and applications. The test covers most of the areas of knowledge and skill listed in Table 14.2 and provides different types of scores, ranging from a norm-referenced total test score to scores on individual items that can be used as program-planning aids. An extensive list of instructional objectives corresponding to the items on the test is included, making it easy to program instruction according to a student's performance on the test.

The Stanford Diagnostic Mathematics Test (Beatty, Maddex, Gardner, & Karlsen, 1984) is another diagnostic instrument. It includes four levels, each designed for administration to a different age group ranging from first grade through high school. Three general areas of skill and knowledge—number system and numeration, computation, and applications—are assessed at each level (these include most of the
<table>
<thead>
<tr>
<th>AREA</th>
<th>EXAMPLES</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basic Information</td>
<td>Number-numeral relationships, counting, equality, symbol names</td>
</tr>
<tr>
<td>Computation skills</td>
<td>Addition, subtraction, multiplication, division</td>
</tr>
<tr>
<td>Problem solving</td>
<td>Writing algorithms for &quot;story problems&quot;</td>
</tr>
<tr>
<td>Fractions</td>
<td>Regular, decimals, percentages, renaming, computation using fractions,</td>
</tr>
<tr>
<td>Fractions</td>
<td>ratios, proportions, probability</td>
</tr>
<tr>
<td>Measurement</td>
<td>Meters and derivatives; inches, feet, miles, etc.; grams and derivatives;</td>
</tr>
<tr>
<td>Geometry</td>
<td>Coin values, equivalencies</td>
</tr>
<tr>
<td>Money</td>
<td>Linear and quadratic equations</td>
</tr>
<tr>
<td>Algebra</td>
<td>Shape names, theorems</td>
</tr>
</tbody>
</table>


Informal inventories are a method of assessing mathematics in which the teacher has the student complete representative examples of different kinds of problems. Informal inventories can be commercially developed or created by teachers and should be aligned with the curriculum in use.

Informal Inventories

Most teachers have students with widely differing skill levels in their classes and must teach each one how to solve different types of problems. Teachers can determine which kinds of mathematics problems are appropriate for each student by using informal inventories. These inventories can be particularly useful to the teacher of students with learning disabilities.

Informal inventories should include representative examples of different kinds of problems. They may be commercially developed or created by teachers themselves. Many of the diagnostic tests described previously, such as the Buswell-John Test (Buswell & John, 1926), do this in a broad way. One important consideration in choosing a commercial informal inventory or creating one is the extent to which it aligns with the curriculum used with the students to be tested. If an informal inventory does not assess the skills the students are expected to acquire, it will not be very helpful in determining what to teach them.

Other informal inventories assess performance in a more detailed way and allow teachers to make precise decisions about planning programs for students. Figure 14.3 (pages 466-469) shows a placement test for geometry knowledge. Each type of task assessed is associated with a general grade level. The grade levels shown are not considered exact or fixed; they simply provide rough guidance about when certain topics might be taught. Because these tests are designed to assess what a student can do, they are given without time limits of teacher assistance. Their purpose is to allow a teacher to determine what students have learned and what they should be taught next. Teachers can modify inventories such as these so that they align with the curriculum. Because these inventories are fairly comprehensive, they can even guide curricular decisions (Stein, Silbert, & Carnine, 1997). Stein and colleagues have provided explicit instructional procedures for teaching each of the skills assessed in the inventories shown in Figure 14.3.

Error Analysis

Teachers can perform an error analysis, or analyze students' mistaken answers to determine what to teach. Student errors may include the following:

1. **Incorrect fact.** The student consistently recalls a fact incorrectly (e.g., \(7 \times 8 = 57\)).
2. **Incorrect operation.** The student executes the incorrect operation (e.g., consistently performs addition when the operation should be multiplication).
3. **Incorrect execution of procedures.** The student applies the steps to an algorithm incorrectly. The procedure may not be known or may be executed in the wrong sequence, or a necessary step may be omitted (e.g., the steps necessary to execute a long-division problem or subtraction with borrowing).
4. **No pattern errors.** The responses are incorrect but appear to be random.
5. **Combination of incorrect facts and incorrectly employed operations and/or algorithms.** (Mastropieri & Scruggs, 2002a, pp. 164-165)

Teachers can use instructional inventories such as the Diagnostic Inventory of Basic Arithmetic Skills (Enright, 1983) and consult books (e.g., Ashlock, 1994) that provide extensive examples of students' mistaken answers, interpretations of them, and general suggestions for their remediation. For example, when a student's answers indicate that the student has failed to "carry," the teacher can employ manipulative aids, such as using bundles of ten sticks and single sticks, drawing boxes in the answer spaces for problems to prompt students to write only one numeral in each column, and playing games requiring the student to trade many chips of lesser value for a single, more valuable chip (Ashlock, 1994).
<table>
<thead>
<tr>
<th>Grade level</th>
<th>Problem type</th>
<th>Performance indicator</th>
</tr>
</thead>
<tbody>
<tr>
<td>1a</td>
<td>Identify circle.</td>
<td>Mark each circle with X.</td>
</tr>
<tr>
<td>1b</td>
<td>Identify rectangle.</td>
<td>Mark each rectangle with X.</td>
</tr>
<tr>
<td>1c</td>
<td>Identify triangle.</td>
<td>Mark each triangle with X.</td>
</tr>
<tr>
<td>1d</td>
<td>Identify square.</td>
<td>Mark each square with X.</td>
</tr>
<tr>
<td>1e</td>
<td>Identify interior of closed figure.</td>
<td>Tell me when I touch the interior of this figure.</td>
</tr>
<tr>
<td>1f</td>
<td>Identify exterior of closed figure.</td>
<td>Tell me when I touch the exterior of this figure.</td>
</tr>
<tr>
<td>2a</td>
<td>Identify cube.</td>
<td>Mark each cube with X.</td>
</tr>
<tr>
<td>2b</td>
<td>Identify sphere.</td>
<td>Mark each sphere with X.</td>
</tr>
<tr>
<td>2c</td>
<td>Identify cone.</td>
<td>Mark each cone with X.</td>
</tr>
<tr>
<td>2d</td>
<td>Identify the diameter of a circle.</td>
<td>What is a diameter? Put X on each line that is the diameter of a circle.</td>
</tr>
<tr>
<td>3a</td>
<td>Measure perimeter.</td>
<td>Find the perimeter of this square.</td>
</tr>
<tr>
<td>3b</td>
<td>Measure area of rectangle or square.</td>
<td>Find the area of this rectangle.</td>
</tr>
<tr>
<td>3c</td>
<td>Identify pyramid.</td>
<td>Mark each pyramid with X.</td>
</tr>
<tr>
<td>3d</td>
<td>Identify cylinder.</td>
<td>Mark each cylinder with X.</td>
</tr>
<tr>
<td>4a</td>
<td>Define/identify radius.</td>
<td>What is the radius of a circle? Mark each line that is a radius with X.</td>
</tr>
<tr>
<td>4b</td>
<td>Using a compass, construct a circle, when given a radius.</td>
<td>Draw a circle that has a radius of 2 inches. Use a compass.</td>
</tr>
<tr>
<td>4c</td>
<td>Label angles.</td>
<td>For each example, write the name of the angle.</td>
</tr>
<tr>
<td>Grade level</td>
<td>Problem type</td>
<td>Performance indicator</td>
</tr>
<tr>
<td>-------------</td>
<td>--------------</td>
<td>-----------------------</td>
</tr>
<tr>
<td>4d</td>
<td>Define degree/measure angles, using a protractor.</td>
<td>Measure each of the following angles.</td>
</tr>
<tr>
<td>4e</td>
<td>Construct angles, using a protractor.</td>
<td>Construct the following angles.</td>
</tr>
<tr>
<td></td>
<td>90°</td>
<td></td>
</tr>
<tr>
<td>4f</td>
<td>Define/identify right angle.</td>
<td>What is a right angle? Circle each right angle.</td>
</tr>
<tr>
<td>4g</td>
<td>Define/identify acute angle.</td>
<td>What is an acute angle? Circle each acute angle.</td>
</tr>
<tr>
<td>4h</td>
<td>Define/identify obtuse angle.</td>
<td>What is an obtuse angle? Circle each obtuse angle.</td>
</tr>
<tr>
<td>4i</td>
<td>Define/identify right triangle.</td>
<td>What is a right triangle? Circle each right triangle.</td>
</tr>
<tr>
<td>4j</td>
<td>Define/identify equilateral triangle.</td>
<td>What is an equilateral triangle? Circle each equilateral triangle.</td>
</tr>
<tr>
<td>4k</td>
<td>Define/identify isosceles triangle.</td>
<td>What is an isosceles triangle? Circle each isosceles triangle.</td>
</tr>
<tr>
<td>4l</td>
<td>Define/identify scalene triangle.</td>
<td>What is a scalene triangle? Circle each scalene triangle.</td>
</tr>
<tr>
<td>4m</td>
<td>Identify the following polygons: pentagon, hexagon, octagon.</td>
<td>Draw a P over the pentagon.</td>
</tr>
<tr>
<td>4n</td>
<td>Measure the volume of a cube.</td>
<td>What is the volume of a figure that is 5 inches long, 3 inches wide, and 6 inches high?</td>
</tr>
<tr>
<td>5a</td>
<td>Identify parallel lines.</td>
<td>Circle each group of parallel lines.</td>
</tr>
<tr>
<td>5b</td>
<td>Identify perpendicular lines.</td>
<td>Circle each group of perpendicular lines.</td>
</tr>
<tr>
<td>5c</td>
<td>Identify a parallelogram.</td>
<td>Circle each parallelogram.</td>
</tr>
</tbody>
</table>


Monitoring Progress

Some of the assessment instruments described in the preceding section may be readministered to determine if a student is making progress. For example, an achievement test may be given in the fall and spring of each year to assess how much progress students are making. Most formal standardized instruments, however, are not designed to be readministered more than once a year. They are not fine-grained enough to be sensitive to small amounts of student progress, and they do not have enough test items at each level to be readministered frequently. If students take these tests repeatedly, they may begin to answer items correctly, not because they have learned the skills, but because they have become familiar with the items. Furthermore, a formal test often takes over 30 minutes, making frequent testing excessively time consuming.

One appropriate means for evaluating progress is to assess students' performances on curricular materials. This approach has many advantages, not the least of which is that the teacher learns how well students are doing on the materials they are using.
What Interventions Help Students with Mathematics Difficulties?

Effective instruction has several features: (a) It takes place in groups, (b) it is teacher directed, (c) it is academically focused, and (d) it is individualized (Stevens & Rosenblith, 1981, p. 1). The Missouri Mathematics Program typifies this approach. It devotes nearly the entire arithmetic period to working on arithmetic problems, provides a daily review, demonstrates new skills, and offers extended opportunities to practice the new skills under individualized teacher supervision and correction. Students using this program learn significantly more than students in traditional programs (Good & Grouws, 1979; Good, Grouws, & Elmore, 1983). Ms. Shein, Shannon's special education teacher, explains some benefits:

Students like Shannon really benefit from demonstration and practice. It's important to present a variety of examples and explain in a step-by-step, think-aloud fashion, how I approach and solve each problem. I have high expectations for my students, and they know that. They want my feedback. They're eager to know if they're on the right track.

See the Effective Practices box on page 472 for other features of effective instruction.

Unfortunately, many math programs in use today are not considered of students with mathematical problems. For example, in teaching fractions, they

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[Table and diagram of add combinations with examples provided]
Prevention and Intervention of Mathematics Difficulties

Researchers have derived four principles of prevention and intervention in the area of mathematics. Each principle represents the following concepts (1) its effectiveness has been demonstrated in math; (2) research supports its use with students with learning disabilities; and (3) research supports its use in general education settings (Fuchs & Fuchs, 2001). The four principles, along with teacher behaviors and student outcomes, are outlined here:

* Principle 1: Quick Pace, Varied Activities, and Engagement
  - The effective teacher incorporates a variety of instructional formats.
  - The effective teacher uses a routine of grouping arrangements.
  - In doing so, students spend almost 100% of the lesson discussing, writing, computing, and problem solving, rather than spending time in slowdown or unnecessary waiting.

* Principle 2: Challenging Achievement Standards
  - The effective teacher devotes time to motivating students by communicating high expectations.
  - In doing so, the students are more likely to be eager to learn, be highly engaged, and perform at higher levels.

In addition to the above principles, Fuchs, Fuchs, and their research team at Vanderbilt University have developed Peer-Assisted Learning Strategies (PALS) in math and reading for students with and without learning disabilities in general education settings. This effective, research-based intervention includes carefully designed elements of both instructional and behavioral principles, including a strong motivational system, quick paced, varied activities, and high levels of engagement between student self-mathematics representations and problem-solving situations. (Fuchs & Fuchs, 2000)

Few teachers have the time to design and implement a program such as the Missouri Mathematics Program on their own. Instead, they must depend on programs adopted by state or local education agencies. Teachers usually must adapt programs to make them suitable for students with learning disabilities. Adaptation requires modifying teaching behaviors (e.g., scheduling plenty of time for mathematics lessons) as well as instructional programming techniques (e.g., sequencing of lessons). The Multicultural Considerations box on page 474 provides suggestions for working with culturally and linguistically diverse students to develop their math skills.

Developmental Interventions

Major publishing companies offer developmental programs in mathematics. Among the most frequently used are Scott, Foresman's Exploring Mathematics (1991), Houghton Mifflin's The Mathematics Experience (1992), and Addison-Wesley's Explorations (1991) series. (From the titles of these programs, one can get an idea of how publishers position their products to stress popular themes; notice the emphasis on exploration, with a hint of discovery, in these titles.)

Developmental programs introduce basic skills such as addition, subtraction, multiplication, and division. Most also introduce other important content areas, such as place value, measurement, geometry, and fractions. Despite the many similarities in what they cover, basal programs differ markedly in when they present material. For example, during the first grade, one program may devote a unit to fractions that may wait until a later grade to do so.

Connecting Math Concepts (Engelmann et al., 1991) is a basal program designed for use in primary through elementary school and is based on the Direct Instruction (DI) model. In highly structured lessons involving frequent teacher questions and student answers, students learn fundamental skills for solving mathematical problems. One important feature of this approach is that teachers explicitly teach students strategies to use in solving problems.

The forerunner of Connecting Math Concepts was DISTAR Arithmetic (Engelmann & Carnine, 1975, 1976). Students whose teachers used DISTAR Arithmetic had higher levels of achievement in arithmetic than those in any of eight other model programs evaluated in one large national study. Not only did the students excel in basic skills such as computation, but they also scored higher on tests of mathematical concepts and problem solving (Abt Associates, 1976, 1977). Similar results have been obtained in comparisons of Connecting Math Concepts with other curricula (Vreeland et al., 1994; Wellington, 1994). Students taught using the DI program not only learn more computation skills, but also learn more concepts.

Factual Arithmetic program (Stern & Stern, 1971) is designed to develop students' understanding of arithmetic principles by giving them extensive experience with manipulating objects. The program is designed for use in kindergarten through third grade and includes different colored blocks and sticks that represent numbers from 1 to 10. The 1 block is a cube, the 3 stick is the equivalent of laying three 1's in a line, and the 10 is the equivalent of laying ten 1's (or two 5's) in a line. Thus, numerical relationships are represented by different lengths. (Similar blocks are used in other approaches, particularly the Cuisenaire rods advocated by Gattegno, 1963).
Ten Ways to Enhance the Math Skills of Culturally and Linguistically Diverse Students

Dixon, 1994; Kame'enui & Carnine, 1998). These features are probably especially important for adolescents with learning disabilities (Jones, Wilson, & Bhowjani, 1997).

The Computational Arithmetic Program, or CAF (Smith & Lovitt, 1982), is designed for use with students who need to learn basic addition, subtraction, multiplication, or division of whole numbers. The program includes directions for monitoring progress, suggestions and materials for reinforcing progress, and an extensive set of carefully sequenced worksheets. It is based on what Smith and Lovitt learned from their research with students who have learning disabilities (Lovitt & Smith, 1978; Smith & Lovitt, 1975, 1976, Smith, Lovitt, & Kidder, 1972).

The Corrective Mathematics program (Engelmann & Carnine, 1981) is composed of several modules, each of which covers a specific area of mathematics skill. These areas include addition, subtraction, multiplication, division, money, and measurement. The program includes scripts for daily lessons, and accompanying workbooks provide students with extensive opportunities for practice. Programs similar to Corrective Mathematics are Fractions (Engelmann & Steely, 1980) and Ratios and Equations (Engelmann & Steely, 1981).

Technology

Arithmetic and mathematics lend themselves well to the use of technology. Their regularity and the value of drill and practice in learning computational skills probably contribute to the popularity of using technology in arithmetic instruction. People may also simply associate computers with mathematics and hence consider arithmetic a "natural" area for applying computer technology.

Calculators

One of the first questions many teachers have about technology and arithmetic is whether it is advisable for students to use calculators. Calculators are common in classrooms today and provide many new learning opportunities. For example, some states provide graphing calculators for students so they can learn how to use them to solve problems. Students with disabilities benefit from using calculators in solving computation problems (Horton, Lovitt, & White, 1992). Of course, teachers must teach their students how to use the calculators. "What remains for many practitioners and researchers, however, is the question of exactly how calculators should complement or substitute for paper-and-pencil practice" (Woodward & Montague, 2002, p. 95). Shannon's father voices his concerns:

I think calculators are great, but I was afraid that today's students would become so addicted and rely too heavily on them. Some use them for basic facts. I remember how hard I worked to learn my multiplication facts when I was young. I wonder if teachers have stopped expecting students to memorize multiplication facts, but Shannon's teacher reassured me, they explained that, even as high school teachers, they continue to work with their students on learning basic facts and mental math skills.

Diane Ireland, Shannon's father

Remedial Interventions

Remedial programs should have the same characteristics as effective developmental programs. Features such as introducing new concepts systematically, providing adequate practice and review, and teaching big ideas are critical (Carnine, Jones, &
Ten Tips for Choosing Math Software

1. The less clutter on the screen, the better.
   Most students with LD are distracted by too much visual information. Choose programs that use simple screens.

2. Use software that matches the student's level.
   Many students with LD get confused if the same task is presented in different ways. Choose software that is appropriate for the student's math level.

3. Choose software that is helpful.
   Math software should provide feedback to the correct answer when a student makes an error. The feedback should indicate the range within which the answer lies. Software should also provide a diagram to indicate the underlying concepts that cause the student to make the problem on their own.

4. Choose software with good record-keeping capabilities.
   Instructional software is a tool in effective math instruction and learning. With color graphics, animation, sound, and interactivity, it can capture and hold the attention of students so that they persist in mathematics tasks. It is important, however, to combine direct teacher instruction with technology-assisted instruction. In most instances, concept development with concrete materials and clear procedural instruction should precede software use.

Computer-Assisted Instruction

Teachers who see a natural fit between mathematics and computers may find some support in evidence about drill-and-practice programs. Drill and practice refers to the repeated presentation of and response to a limited number of facts. Although drill-and-practice programs are under attack as undesirable, facility with the automatic retrieval of facts is important in solving problems. Computers can provide the practice opportunities for many students with learning disabilities need to master basic facts (Hasselbring, Goin, & Brassard, 1987).

Computers can also be useful in teaching more conceptual arithmetic content. Instructional programs such as Mastering Fractions (Systems Impact, 1986) and Mastering Equations, Roots, and Exponents (Systems Impact, 1988) illustrate the application of technology to learning more than basic facts. Examples of concepts and operations are presented on videodisks, and colorful, animated demonstrations show how, for example, to determine whether a fraction is less than, equal to, or greater than 1 or how to subtract fractions with unlike denominators. While circulating among the students, the teacher controls the videodisk with a remote control device, and the students answer questions posed by the presenters on video monitors showing the program. As is typical of Direct Instruction programs, the questions are rapid, the demonstrations change in subtle but important ways, and the students answer specific questions either chorally or in writing.

The Mastering Fractions program produces substantially greater achievement by students with learning disabilities and those in remedial education than do other chart, or fraction strips can give the student tools to represent a given problem and then go on to solve it. These virtual manipulations are incorporated in such programs as Equivalent Fractions by Sunburst (www.sunburst.com).

Select software that simulates real-life situations.
In real life, there is usually more than one way to solve a problem. Money, time, and problem-solving software is more effective if it allows multiple roads to problem situation. Making Change by Attainment Company, Inc. (www.attainment-ine.com), for example, combines decisions with multiple solution choices.

Remember, software is a learning tool—not the total solution.
Instructional software is a tool in effective math instruction and learning. With color graphics, animation, sound, and interactivity, it can capture and hold the attention of students so that they persist in mathematics tasks. It is important, however, to combine direct teacher instruction with technology-assisted instruction. In most instances, concept development with concrete materials and clear procedural instruction should precede software use. Pencil and paper tasks still have a role in student learning (Babbitt, 1999).
published instructional programs that cover the same content. In fact, this program helps students master about the same percentage of the objectives for instruction in prealgebra as their non-LD peers master in general education (Grosen & Ewing, 1994; Kelly, Carnine, Gersten, & Grosen, 1986; Lukeke, Roger, & Evans, 1989). The Today’s Technology box on pages 476-477 provides tips for choosing math software. The instructional technology of such programs is more important than the computer technology used in them. One team of researchers compared the effectiveness of the Mastering Fractions program when it was presented using the videodisk technology to the same program when a teacher presented the lessons according to the scripts but without the colorful animation (Hasselbring, Sherwood, & Brunsfield, 1986). Although teachers preferred to use the videodisk version (it was easier for them), student achievement did not differ under the two conditions. Similar results from other studies (e.g., Gleason, Carnine, & Boriero, 1990) support the idea that the means of delivering instruction is not the critical variable in raising the achievement of students with learning disabilities.

Video technology can also be used to simulate authentic context in which students use mathematics knowledge. Shannon’s math teacher explains:

I’ve gotten some good ideas from research studies I’ve read by Bottge and others (e.g., Bottge, 2002a; Bottge, Heinrich, Chair, & Sordahl, 2001). They’ve conducted studies where students work on video-based math problems... the videos present realistic scenarios, and the students work in groups to solve the problems. I’ve had great success with this approach. I explicitly tell the students the skills they’ll need, and then, they apply what they’ve learned by solving the problems presented via video. The students identify with the individuals in the videos, and they’ve been much more motivated.

Effective Teaching Procedures

To benefit students with learning disabilities, beginning math instruction should focus, in part, on developing students’ number sense (Gersten & Chard, 1999). Research-based strategies for teaching mathematics include modeling, reinforcement, strategy training, and self-instruction training (see Table 14.3).

Modeling

A mainstay in teaching, modeling may be used in any of several ways:

• The teacher demonstrates for the student (e.g., “Watch me; here’s how I do a division problem”).
• The teacher has another student demonstrate (e.g., “Watch Judy; she’s going to bisect that line”).

Table 14.3. Algebra Instruction for Students with Learning Disabilities

<table>
<thead>
<tr>
<th>INSTRUCTIONAL FEATURES</th>
<th>EFFECTIVE TEACHING BEHAVIORS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Before</td>
<td>Administer daily quiz of previously learned skills (Kitz &amp; Thorpe, 1995).</td>
</tr>
<tr>
<td>Review</td>
<td>Discuss with individual or in group to introduce the strategy; explain difficulties with problem-solving; explain rationale of self-questioning strategies for problem representation and solution of various problem types (Huntington, 1994; Hutchinson, 1989, 1993).</td>
</tr>
<tr>
<td>General orientation</td>
<td>Provide clear and precise presentation of the concept, skill, and/or strategy using a wide range of examples and nonexamples and manipulatives and computer-assisted instruction (Huntington, 1994; Hutchinson, 1989, 1993; Kitz &amp; Thorpe, 1995).</td>
</tr>
<tr>
<td>Teacher modeling</td>
<td>Direct students to important problem-solving steps via structured worksheets (Huntington, 1994; Hutchinson, 1989, 1993).</td>
</tr>
<tr>
<td>Guided practice</td>
<td>Teach students to ask themselves questions during problem representation and solution (Huntington, 1994; Hutchinson, 1989, 1993).</td>
</tr>
<tr>
<td>Feedback and reinforce</td>
<td>Provide opportunities for guided practice, including a wide range of examples and nonexamples (Huntington, 1994; Hutchinson, 1989, 1993).</td>
</tr>
<tr>
<td>After</td>
<td>Provide extensive cumulative reviews (Kitz &amp; Thorpe, 1995).</td>
</tr>
<tr>
<td>Cumulative reviews</td>
<td>Show and discuss graph of student progress (Huntington, 1994; Hutchinson, 1989, 1993).</td>
</tr>
</tbody>
</table>


What Interventions Help Students with Mathematics Difficulties?